REPRESENTATIONS OF EPI-LIPSCHITZIAN SETS

MARC-OLIVIER CZARNECKI AND ANASTASIA NIKOLAEVNA GUDOVICH

Abstract. A closed subset M of a Banach space E is epi-Lipschitzian, i.e., can be represented locally as the epigraph of a Lipschitz function, if and only if it is the level set of some locally Lipschitz function $f : E \to \mathbb{R}$, wich Clarke's generalized gradient does not contain 0 at points in the boundary of M, i.e., such that:

$$M = \{x \mid f(x) \leq 0\},$$
$$0 \notin \partial f(x) \text{ if } x \in \mathrm{bd}\, M.$$

This extends the characterization previously known in finite dimension and answers to a standing open question

1. Introduction and results

A subset M of a Banach space E is epi-Lipschitzian (or epi-Lipschitz) at a point $x \in M$ if it can be represented as the epigraph of a Lipschitz function in some neighborhood of x. Precisely

Definition 1. *Let M be a subset of a Banach space E, and let $x \in M$. The set M is epi-Lipschitzian at x if and only if there exists a neighborhood U of x, a Banach space F, a Banach isomorphism $A : E \to F \times \mathbb{R}$ and a map $\Phi : F \to \mathbb{R}$ which is Lipschitz around the F component of $A(x)$, such that*

$$M \cap U = U \cap A^{-1}(epi\Phi),$$

where $epi\Phi = \{(\xi, t) | \Phi(\xi) \leq t\}$.

The set M is epi-Lipschitzian if it is epi-Lipschitzian around every $x \in M$.

Rockafellar [8] formally introduces the notion of epi-Lipschitzian sets in Banach spaces, and precisely gives the above definition in [9], in finite dimension. We stick to this definition which is meant by the word "epi-Lipschitzian", epigraph of Lipschitz function. The definition can also be given at a point $x \notin M$ but then requires local closedness of M to work with.

The purpose of our paper is to give a direct and self content proof of the characterizion of epi-Lipschitzian sets as level sets of Lipschitz functions satisfying a nondegeneracy condition, thus extending results previously known in finite dimension.

One of our purposes, besides answering to a standing open question, is to provide tools for the existence of fixed points and equilibria in the non convex case. In finite dimension, representation as level sets permits to use an approximation method and to obtain finer results ([4, Theorems 2.2 and 2.3]) than with direct methods. In infinite dimension, epi-Lipschitzian sets already play a major role ([1, 6]) in the fixed point theory. Of course, since non empty epi-Lipschitzian sets have non empty interiors, they are not compact in infinite dimension and one expects extension of Schauder Theorem, and certainly not of Fan-Browder theorem.

Before stating our main result, we recall the definition of Clarke's generalized gradient. Let $f : E \to \mathbb{R}$ be locally Lipschitz, the generalized directional derivative of f at $x \in E$ in the direction $v \in E$ (see [2]) is defined by:

$$f^0(x,v) = \limsup_{y \to x, t>0, t \to 0} \frac{f(y+tv) - f(y)}{t}.$$

The generalized gradient of f at x is the set:

$$\partial f(x) = \{\xi \in E^* \mid \forall v \in X,\ f^0(x,v) \geq \langle \xi, v \rangle\}.$$

Theorem 1. *(global representation) Let M be a closed subset of a Banach space E. The two following assertions are equivalent:*
(i) M is epi-Lipschitzian,
(ii) there is a locally Lipschitz function $f : E \to \mathbb{R}$ such that:

$$\begin{array}{lll} (rep) & \textit{(set representation)} & M = \{x \mid f(x) \leq 0\}, \\ (nd) & \textit{(nondegeneracy)} & 0 \notin \partial f(x) \textit{ if } x \in \operatorname{bd} M. \end{array}$$

The proof of the implication $(i) \Rightarrow (ii)$ is given by Cwiszewski and Kryszewski [5] with the signed distance function Δ_M, defined by

$$\Delta_M = d_M - d_{E \setminus M},$$

where d_M is the distance function to the set M. The function Δ_M is Lipschitz with constant one, and Theorem 1 holds considering globally Lipschitz functions instead of locally Lipschitz functions.

Theorem 2. [5, Proposition 3.10] *If $M \subset E$ is epi-Lipschitzian, then for all $x \in \operatorname{bd} M$, $0 \notin \partial \Delta_M(x)$.*

The implication $(ii) \Rightarrow (i)$ is given in Section 2.

Remark 1.1. Theorem 1 generalizes to the infinite dimension the corresponding finite dimensional result. The proof of the remaining implication $(i) \Rightarrow (ii)$ is then given by [3, Proposition 4.1] which [5, Proposition 3.10] generalizes to the infinite dimension.

The classical way to prove $(ii) \Rightarrow (i)$ is as follows. Take $x \in \operatorname{bd} M$. From the continuity of f we get $f(x) = 0$. Then, since $0 \notin \partial f(x)$, Clarke [2, Corollary 1, p. 56] shows that

$$N_M(x) \subset \cup_{\lambda \geq 0} \lambda \partial f(x),$$

where $N_M(x)$ denotes Clarke's normal cone to M at $x \in M$. It implies that $N_M(x)$ is pointed (i.e $N_M(x) \cap -N_M(x) = \{0\}$). But in finite dimension, Rockafellar [9] characterizes closed epi-Lipschitzian sets as sets having pointed normal cone. This is no longer true in infinite dimension, and Rockafellar [9] gives the following counterexample:
Let M be the closed convex subset of the Hilbert space $l^2 \times \mathbb{R}$ which is the epigraph of the function

$$\varphi(\xi) = \sum_{j=1}^{\infty} j \xi_j^2, \textit{ where } \xi = (\xi_1, \xi_2, ...).$$

M is not epi-Lipschitzian (it has an empty interior), but

$$N_M(0) = \{0_{l^2}\} \times \mathbb{R}^-,$$

which is clearly pointed. Thus we cannot follow this path to prove the implication $(ii) \Rightarrow (i)$ in infinite dimension.

Remark 1.2. Theorem 1 gives an answer to questions raised by Bader and Kryszewski [1], and by Cwiszewski and Kryszewski [6]. In [1], the authors raise the question of equivalence of the then following assertions, taking M to be a closed subset of a Banach space E :

$$\begin{array}{lll} (i) & M \text{ is epi-Lipschitzian;} & \\ (ii) & \forall x \in \operatorname{bd} M, & N_M(x) \text{ is pointed;} \\ (iii) & \forall x \in \operatorname{bd} M, & 0 \notin \partial\Delta_M(x) \text{ and } T_M(x) = \partial\Delta_M(x)^-, \end{array}$$

where $T_M(x)$ denotes Clarke's cone to M at $x \in M$ and $\partial\Delta_M(x)^-$ is the polar cone to the (Clarke) generalized gradient of Δ_M at x.

They notice the implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (ii)$. They prove the implication $(i) \Rightarrow (iii)$ and wonder if the implications $(ii) \Rightarrow (i)$, $(ii) \Rightarrow (iii)$ hold. The answer is no, in view of the counterexample of Rockafellar. But the implication $(iii) \Rightarrow (i)$ holds, from Theorem 1. In [6, Example 3.2], the authors take Assumption (ii) to be the definition of an epi-Lipschitzian set. Theorem 1 shows that this is consistent with the classical definition.

Remark 1.3. It is sometimes necessary to consider sets which are only epi-Lipschitzian on some part. Theorem 1 can be thus formulated more generally as follows.

Theorem 3. *(local representation) Let M be a subset of a Banach space E, and let $M_L \subset M$. The two following assertions are equivalent:*
(i) M is epi-Lipschitzian on M_L, i.e., at every point of M_L;
(ii) there is an open subset U containing M_L and a locally Lipschitz function $f : U \to \mathbb{R}$ such that:

$$\begin{array}{lll} (rep) & \textit{(set representation)} & M \cap U = \{x \in U \mid f(x) \leq 0\}, \\ (nd) & \textit{(nondegeneracy)} & 0 \notin \partial f(x) \textit{ if } x \in \operatorname{bd} M \cap U. \end{array}$$

We let the reader check that the proof of Theorem 1 (the proof given below and the proof of [5, Proposition 3.10]) allows to prove Theorem 3, and that, with an Urysohn type argument, Theorem 3 implies Theorem 1.

2. Proof of Theorem 1

In this section, we prove the implication $(ii) \Rightarrow (i)$. Our proof is a nontrivial adaptation of the finite dimensional proof of [3, Proposition 4.4]. In this paper, the proof was "left to the reader." The same result appears in [7] as Proposition 5.1, page 52, but the proof makes unnecessary use of the finite dimensional structure, among which the scalar product.

For an element $v \in E$ and a subset U of E, we define (as in [7] and [3]) the function $\lambda_{v,U} : E \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$\lambda_{v,U}(x) = \inf\{t \mid x + tv \in M \cap U\}.$$

We assume (ii) and consider $x \in \operatorname{bd} M$. We now find sets U and V, neighborhoods of x, such that the function $\lambda_{v,U}$ is real valued and Lipschitzian on V, which allows us to write $M \cap V$ as the epigraph of $\lambda_{v,U}$ restricted to a certain subspace.

Since $0 \notin \partial f(x)$, by the definition of $\partial f(x)$, and from the positive homogeneity of the generalized directional derivative, there exists $v \in E$ such that $\|v\| = 1$ and

$$f^0(x, v) < 0.$$

Take a real number $\alpha > 0$ such that

$$f^0(x, v) < -\alpha.$$

By definition of the generalized directional derivative $f^0(x, v)$, there exists a real number $r > 0$ such that

$$\forall y \in B(x, 2r), \qquad \forall t \in (-2r, 2r) \setminus \{0\}, \qquad \frac{f(y + tv) - f(y)}{t} < -\alpha. \tag{1}$$

Strictly speaking, the definition of $f^0(x, v)$ yields the above inequality for positive real numbers t, but it is easily deduced for negative t.[1] Without loss of generality, possibly taking a smaller real number $r > 0$, we can assume that the function f satisfies the Lipschitz condition on $B(x, r)$ with a some constant k. Set

$$\varepsilon = inf \left\{ \frac{r}{4}, \frac{\alpha r}{4k} \right\}.$$

Lemma 1. *The function $\lambda_{v,B(x,r)}$ is real-valued on $B(x, \varepsilon)$ and*

$$\forall y \in B(x, \varepsilon), \qquad | \lambda_{v,B(x,r)}(y) | \leq \frac{r}{4}. \tag{2}$$

Proof of Lemma 1. Take $y \in B(x, \varepsilon)$, the set

$$\{t \mid y + tv \in M \cap B(x, r)\}$$

is bounded from below, since the set $M \cap B(x, r)$ is bounded (write $|tv| \leq |y + tv - x| + |x - y| < r + \varepsilon$) and it contains the real number $\frac{r}{4}$, thus showing that $\lambda_{v,B(x,r)}(y) \in \mathbb{R}$.

Indeed, first note

$$y + \frac{r}{4} v \in B(x, r)$$

from the inequality $\|y + \frac{r}{4}v - x\| \leq \|y - x\| + \frac{r}{4} < \varepsilon + \frac{r}{4} < r$. Using (1) we get

$$f\left(y + \frac{r}{4}v\right) - f(y) < -\alpha \frac{r}{4}.$$

But

$$f(y) - f(x) \leq k\|x - y\|$$

and

$$f(x) = 0 \text{ (since } x \in \operatorname{bd} M).$$

Thus

$$f\left(y + \frac{r}{4}v\right) < -\alpha \frac{r}{4} + k\varepsilon \leq 0.$$

Hence

$$y + \frac{r}{4}v \in M$$

and

$$\frac{r}{4} \in \{t \mid y + tv \in M \cap B(x, r)\}.$$

Then $\lambda_{v,B(x,r)}(y) \in \mathbb{R}$ and

$$\lambda_{v,B(x,r)}(y) \leq \frac{r}{4}.$$

Now take a real number $t < 0$ such that $y + tv \in M \cap B(x, r)$. Since $| t | = \|tv\| \leq \|y + tv - x\| + \|y - x\| < r + \varepsilon$, we have $| t | < 2r$. From (1)

$$f(y + tv) - f(y) > -\alpha t.$$

[1] Assume that (1) to be valid for positive t, with constant $4r$. If $-2r < t < 0$, then $y + tv \in B(x, 4r)$, $(f(y + tv - tv) - f(y + tv))/(-t) < -\alpha$, i e., $(f(y + tv) - f(y))/t < -\alpha$.

But

$$f(x) - f(y) \leq k\|x - y\|.$$

Recalling that $f(x) = 0$ (since $\mathrm{bd}\, M \subset \{x \mid f(x) = 0\}$ by continuity of f) and that $f(y + tv) \leq 0$ (since $y + tv \in M$), we deduce $-k\varepsilon - \alpha t \leq 0$. Thus, $t \geq -\frac{r}{4}$ and

$$\lambda_{v,B(x,r)}(y) \geq -\frac{r}{4}.$$

□

From Hahn Banach theorem, there exists a continuous linear map φ such that $\varphi(v) = 1$ and $\|\varphi\| = 1$. Set

$$F = Ker\ \varphi.$$

Then

$$E = F \oplus \mathbb{R}v$$

and F is closed. Define the (continuous) projection on F along v

$$\begin{array}{rcl} \pi\ :\ E & \rightarrow & F \\ y & \mapsto & y - \varphi(y)v \end{array}$$

Lemma 2. *For every $y \in B(x, \varepsilon)$ and $\lambda \in \mathbb{R}$*

$$\lambda_{v,B(x,r)}(y + \lambda v) = \lambda_{v,B(x,r)}(y) - \lambda. \tag{3}$$

Moreover, the function $\lambda_{v,B(x,r)}$ is real valued on the cylinder

$$B_F(\pi(x), \varepsilon) + \mathbb{R}v,$$

where $B_F(a, \varepsilon) = \{b \in F \mid \|b - a\| < \varepsilon\}$, and for every y in the cylinder,

$$y + (\varphi(x) - \varphi(y))v \in B(x, \varepsilon).$$

Proof of Lemma 2. Using definition of $\lambda_{v,B(x,r)}$ we get

$$\begin{array}{rcl} \lambda_{v,B(x,r)}(y + \lambda v) & = & inf\{t \mid y + \lambda v + tv \in M \cap B(x,r)\} \\ & = & inf\{t \mid y + tv \in M \cap B(x,r)\} - \lambda \\ & = & \lambda_{v,B(x,r)}(y) - \lambda. \end{array}$$

Now we must only prove that if $y \in B_F(\pi(x), \varepsilon) + \mathbb{R}v$, then $y + (\varphi(x) - \varphi(y))v \in B(x, \varepsilon)$ which finishes the proof of Lemma 2 in view of (3). Indeed, $y = y_F + tv$, with $y_F \in B_F(\pi(x), \varepsilon)$. By uniqueness of decomposition, we have

$$\begin{array}{rcl} y_F & = & \pi(y) = y - \varphi(y)v \\ t & = & \varphi(y) \end{array}$$

and $\|y - \varphi(y)v + \varphi(x)v - x\| = \|y_F - \pi(x)\| < \varepsilon$. □

Lemma 3. *For every $y \in B_F(\pi(x), \varepsilon) + \mathbb{R}v$*

$$y + \lambda_{v,B(x,r)}(y)v \in \mathrm{bd}\, M \cap B\left(x, \frac{r}{2}\right).$$

Proof of Lemma 3. From the definition of $\lambda_{v,B(x,r)}$ it follows that $\lambda_{v,B(x,r)}(y) = \lim_{n\to\infty} t_n$ with $y + t_n v \in M \cap B(x,r)$, and since the set M is closed, we have

$$y + \lambda_{v,B(x,r)}(y)v \in M.$$

Now we show that

$$y + \lambda_{v,B(x,r)}(y)v \in B\left(x, \frac{r}{2}\right).$$

Indeed,

$$
\begin{aligned}
\|y+\lambda_{v,B(x,r)}(y)v-x\| &= \|y-\varphi(y)v-(x-\varphi(x)v)+(\lambda_{v,B(x,r)}(y)+\varphi(y)-\varphi(x))\| \\
&= \|\pi(y)-\pi(x)+\lambda_{v,B(x,r)}\big(y+(\varphi(x)-\varphi(y))v\big)v\| \\
&< \varepsilon+\mid \lambda_{v,B(x,r)}\big(y+(\varphi(x)-\varphi(y))v\big)\mid . \qquad (4)
\end{aligned}
$$

From Lemma 2 it follows that $y+(\varphi(x)-\varphi(y))v \in B(x,\varepsilon)$, so Lemma 1 implies that

$$\mid \lambda_{v,B(x,r)}(y+(\varphi(x)-\varphi(y))v)\mid \leq \frac{r}{4}.$$

Hence,

$$\|y+\lambda_{v,B(x,r)}(y)v-x<\varepsilon)+\frac{r}{4}\leq\frac{r}{2}.$$

The proof of Lemma 3 is finished by noticing (the converse easily leads to a contradiction) that

$$y+\lambda_{v,B(x,r)}(y)v\notin intM. \quad \square$$

Lemma 4. *The function $\lambda_{v,B(x,r)}$ is Lipschitz on the cylinder $B_F(\pi(x),\varepsilon)+\mathbb{R}v$.*

Proof of Lemma 4. Take y and z in the cylinder. Set

$$y'=y+(\varphi(x)-\varphi(y))v,$$

$$z'=z+(\varphi(x)-\varphi(z))v.$$

From Lemma 2, the elements y' and z' belongs to $B(x,\varepsilon)$. Hence

$$\|z'-y'\|\leq 2\varepsilon,$$

and, in view of Lemma 1,

$$\lambda_{v,B(x,r)}(y')\leq\frac{r}{4}.$$

Consequently,

$$
\begin{aligned}
y'+\left(\lambda_{v,B(x,r)}(y')+\frac{k}{\alpha}\|z'-y'\|\right)v &\in B(x,r),\\
z'+\left(\lambda_{v,B(x,r)}(y')+\frac{k}{\alpha}\|z'-y'\|\right)v &\in B(x,r).
\end{aligned}
$$

Further, since f is k-Lipschitz on $B(x,r)$, we get

$$
\begin{aligned}
(5)\quad & f\left(z'+\left(\lambda_{v,B(x,r)}(y')+\frac{k}{\alpha}\|z'-y'\|\right)v\right)\\
&\qquad -f\left(y'+\left(\lambda_{v,B(x,r)}(y')+\frac{k}{\alpha}\|z'-y'\|\right)v\right)\leq k\|z'-y'\|.
\end{aligned}
$$

The point y' belongs to the cylinder, and by Lemma 3,

$$y'+\lambda_{v,B(x,r)}(y')\in \mathrm{bd}\,M\cap B(x,r).$$

On the other hand, $\frac{k}{\alpha}\|z'-y'\|\leq r$, and by (1) we obtain

$$
\begin{aligned}
(6)\quad & f\left(y'+\left(\lambda_{v,B(x,r)}(y')+\frac{k}{\alpha}\|z'-y'\|\right)v\right)\\
&\qquad \leq f\left(y'+\lambda_{v,B(x,r)}(y')\right)-k\|z'-y'\|\leq -k\|z'-y'\|.
\end{aligned}
$$

Adding (5) and (6) we get

$$f + \left(z' + \left(\lambda_{v,B(x,r)}(y') + \frac{k}{\alpha}\|z' - y'\|\right) v\right) \leq 0.$$

Thus,

$$z' + \left(\lambda_{v,B(x,r)}(y') + \frac{k}{\alpha}\|z' - y'\|\right) v \in M.$$

Since $z' + \left(\lambda_{v,B(x,r)}(y') + \frac{k}{\alpha}\|z' - y'\|\right) v \in M \cap B(x,r)$, from the definition of $\lambda_{v,B(x,r)}$ we get

$$\lambda_{v,B(x,r)}(z') \leq \lambda_{v,B(x,r)}(y') + \frac{k}{\alpha}\|z' - y'\|.$$

But, in view of Lemma 2,

$$\lambda_{v,B(x,r)}(y') = \lambda_{v,B(x,r)}(y) + \varphi(y) - \varphi(x), \tag{7}$$

$$\lambda_{v,B(x,r)}(z') = \lambda_{v,B(x,r)}(z) + \varphi(z) - \varphi(x). \tag{8}$$

Note that

$$\|z' - y'\| \leq \|z - y\| + \mid \varphi(z) - \varphi(x) \mid,$$

and $\mid \varphi(z) - \varphi(x) \mid \leq \|z - y\|$ since $\|\varphi\| = 1$.
Now, subtracting (7) from (8) we obtain

$$\lambda_{v,B(x,r)}(z) \leq \lambda_{v,B(x,r)}(y) + \left(1 + \frac{2k}{\alpha}\right) \|z - y\|.$$

The converse inequality is proved by exchanging y and z. □

Lemma 5.

$$\begin{aligned} M \cap B(x,\varepsilon) &= \{y \in B(x,\varepsilon) \mid \lambda_{v,B(x,r)}(y) \leq 0\} \\ &= \{y \in B(x,\varepsilon) \mid \lambda_{v,B(x,r)}(y - \varphi(y)v) \leq \varphi(y)\}. \end{aligned}$$

Proof of Lemma 5. In view of Lemma 2,

$$\lambda_{v,B(x,r)}(y - \varphi(y)v) = \lambda_{v,B(x,r)}(y) + \varphi(y)$$

and we only prove the first equality.
Take. $y \in M \cap B(x,\varepsilon)$, then

$$0 \in \{t \mid y + tv \in M \cap B(x,r)\}$$

and $\lambda_{v,B(x,r)}(y) \leq 0$.
Conversely, take $y \in B(x,\varepsilon)$ such that $\lambda_{v,B(x,r)}(y) \leq 0$. From Lemma 1, $\mid \lambda_{v,B(x,r)}(y) \mid \leq \frac{r}{4}$ and $y + \lambda_{v,B(x,r)}(y)v \in B(x,r)$. Applying (1) at the point $y + \lambda_{v,B(x,r)}(y)v$ we get

$$f(y) \leq f(y + \lambda_{v,B(x,r)}(y)v) - \alpha(-\lambda_{v,B(x,r)}(y)) \leq 0.$$

But, by definition of $\lambda_{v,B(x,r)}(y)$ -which, in view of Lemma 1 is real valued since $y \in B(x,\varepsilon)$- and since the set M is closed, $y + \lambda_{v,B(x,r)}(y)v \in M$ and $f(y + \lambda_{v,B(x,r)}(y)v) = 0$. Thus $f(y) \leq 0$, i.e., $y \in M$. □

We define the linear map

$$\begin{aligned} A : E &\rightarrow F \times \mathbb{R} \\ y &\mapsto (y - \varphi(y)v, \varphi(y)). \end{aligned}$$

Clearly, A is continuous, invertible map and the inverse map A^{-1}, which is given by $A^{-1}(x,t) = x + tv$, is continuous.
Let $\Phi : F \to \mathbb{R}$ be a Lipschitz function such that for all $y \in B_F(\pi(x), \varepsilon)$,

$$\Phi(y) = \lambda_{v,B(x,r)}(y).$$

The proof of the implication $(ii) \Rightarrow (i)$ of Theorem 1 is finished with the following lemma.

Lemma 6. $M \cap B\left(x, \frac{\varepsilon}{2}\right) = B\left(x, \frac{\varepsilon}{2}\right) \cap A^{-1}(epi\Phi)$.

Proof of Lemma 6. It is a straightforward consequence of Lemma 5 and of the definition of the linear map Φ, if we notice that, for every $y \in B(x, \frac{\varepsilon}{2})$

$$\|\pi(y) - \pi(x)\| = \|y - \varphi(y)v - (x - \varphi(x)v)\| \leq 2\|x - y\| < \varepsilon.$$

Further, if $y \in M \cap B(x, \frac{\varepsilon}{2})$, then Lemma 5 implies that $\lambda_{v,B(x,r)}(\pi(y)) \leq \varphi(y)$. Since $\pi(y) \in B_F(\pi(x), \varepsilon)$ then $\Phi(y) = \lambda_{v,B(x,r)}(\pi(y))$. Hence,

$$A(y) = (\pi(y), \varphi(y)) \in epi\Phi,$$

i.e. $y \in A^{-1}(epi\Phi)$.
Conversely, if $y \in B(x, \frac{\varepsilon}{2}) \cap A^{-1}(epi\Phi)$, then $\pi(y) \in B_F(\pi(x), \varepsilon)$ and $\lambda_{v,B(x,r)}(\pi(y)) = \Phi(\pi(y)) \leq \varphi(y)$. Hence, $y \in M$ by Lemma 5. $\square$

References

[1] Bader, Ralf; Kryszewski, Wojciech On the solution sets of differential inclusions and the periodic problem in Banach spaces. Nonlinear Anal. 54 (2003), no. 4, 707–754.

[2] F.Clarke, Optimisation and nonsmooth analysis, Wiley, New York, 1983.

[3] B. Cornet, M.-O. Czarnecki, Smooth representations of epi-Lipschitzian subsets , Nonlinear Anal. 37 (1999), no. 2, Ser. A: Theory Methods,139–160.

[4] Cornet, Bernard; Czarnecki, Marc-Olivier Existence of generalized equilibria. Nonlinear Anal. 44 (2001), no. 5, Ser. A: Theory Methods, 555–574.

[5] Cwiszewski A., Kryszewski W., Equilibria of set-valued maps: a variational approach, Nonlinear Anal. 48 (2002), no. 5, 707–746.

[6] Cwiszewski, Aleksander; Kryszewski, Wojciech The constrained degree and fixed-point index theory for set-valued maps. Nonlinear Anal. 64 (2006), no. 12, 2643–2664.

[7] Czarnecki, Marc-Olivier, thèse de doctorat, Université Paris 1 Panthéon Sorbonne, 1996.

[8] Rockafellar, R. T. Generalized directional derivatives and subgradients of nonconvex functions. Canad. J. Math. 32 (1980), no. 2, 257–280.

[9] R. T. Rockafellar, Clarke's tangent cones and the boundaries of closed sets in R^n, Nonlinear Anal., Theory, Methods & Aplications, 3 (1979), no. 1, 145–154.

Institut de Mathematiques et Modelisation de Montpellier UMR 5030 CNRS - Universite Montpellier 2 case courier 51, Universite Montpellier 2 Place Eugene Bataillon, 34095 Montpellier Cedex 5

E-mail address: marco@univ-montp2.fr